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AFOSR TN-58-749
ASTIA DOCUMENT NO. AD-201-501

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. BR-27

Transcendental Equations in Electromagnetic Theory

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Mathematics Division
Air Force Office of Scientific Research
Contract No. AF 49(638)-229
Project No. 47500

AUGUST, 1958

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Abstract

The problem of determining the zeros of the Hankel function $H_\nu^1(x)$ for fixed real x and variable (complex) ν plays a role in propagation theory and in the theory of diffraction by a metallic sphere. The equation $H_\nu^1(x) = 0$ defines ν as a function of x. The qualitative behavior of this function (for $x > 0$) is determined for all $x \geq \frac{1}{4}$. Complex values of x are also considered. In the case of a sphere of finite conductivity, the corresponding transcendental equation is similarly treated.

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1. Introduction

The theory of diffraction of electromagnetic waves by a sphere requires a knowledge of the zeros of certain transcendental functions.

In the simplest case of a perfectly conducting sphere of radius a, the equation in question is $H_\nu^1(ka) = 0$, where H_ν^1 is the Hankel function of the first kind of order ν, and k is the wave number of the incident wave. The quantity to be determined is the order ν, with ka being regarded as a parameter. Special attention is paid to the case in which ka is real.

Essentially, then, the question is: Given z, for which values of ν is $H_\nu^1(z) = 0$? While we do not answer this question explicitly, we do investigate the behavior of ν as a function of the argument. Specifically, we derive certain inequalities satisfied by the real and imaginary parts of ν and the argument z; we investigate the behavior of ν as $z \to 0$; we obtain a workable expression for $\frac{d\nu}{dz}$, and we find the approximate location of the large zeros.

A more complicated function must be considered when the conductivity of the sphere is finite. This function, W_ν, is the Wronskian of H_ν^1 and J_ν. The argument of H_ν^1 depends on the ambient dielectric constant, and that of J_ν on the dielectric constant of the sphere. The techniques used in the case of the perfectly conducting sphere are not so applicable here because much less is known of W_ν than of H_ν^1. However, we do find that the large zeros of the Wronskian behave in the same manner as those of the Hankel function. In addition we obtain a series of conditions under which W_ν, does not vanish.

Although many of the results can be extended, the argument z of H_ν^1 will be restricted for the most part to the right half plane; i.e., $|\arg z| \leq \frac{\pi}{2}$. This represents the physically significant case.

As usual, a bar will denote the conjugate complex of a quantity. Because of the relation $\overline{H_\nu^1(z)} = H_\nu^2(\bar{z})$, any result concerning a zero of H_ν^1 can be interpreted in terms of H_ν^2 ; e.g., for real z, the zeros of H_ν^1 and H_ν^2 are conjugates.

Another immediate result is the fact that the zeros of H_ν^1 are symmetric about the origin, since $H_{-\nu}^1(z) = e^{i\nu\pi} H_\nu^1(z)$, so there is no loss of generality in letting the real part of the zeros be non-negative, as we shall sometimes do.

Throughout this paper, the letters x and y will be reserved for the real and imaginary parts of z, and $\alpha + i\beta$ will likewise denote ν.

2. The order of growth of $H_\nu^1(z)$ with respect to ν

The preceding results, and all the subsequent ones predicated on the vanishing of H_ν^1, would be true vacuously if there were no zeros. To lend substance to these results, therefore, it is necessary to prove the existence of zeros. For this purpose we shall apply function theoretical considerations. Specifically, since H_ν^1 is not only analytic but entire in ν, we shall apply the concept of the order of an entire function to prove that there exists an infinite number of zeros, and also to obtain an infinite product representation.

From Heine's formula (see [1], p. 26), for $0 < \arg z < \pi$,

$$g(\nu) = \frac{\pi}{2} e^{(\nu + 1)\pi i/2} H_\nu^1(z) = \int_0^\infty e^{iz \cosh t} \cosh \nu t \, dt.$$

If we let $|\nu| = \rho$,

$$|g(\nu)| \leq \int_0^\infty e^{-y \cosh t + \rho t} \, dt.$$

We shall show that the order of $g(\nu)$ is ≤ 1; i.e., that

$$|g(\nu)| < Me^{\rho^{1+\epsilon}} \quad \text{for all } \epsilon > 0, \text{ as } \rho \to \infty,$$

where M is a suitable quantity which may depend on ϵ but is independent of ρ. It suffices to show that

$$I \equiv \int_0^\infty e^{-y \cosh t + \rho(t - \rho^\epsilon)}\, dt < M.$$

But

$$I \leq \int_0^{\rho^\epsilon} e^{-y \cosh t}\, dt + \int_{\rho^\epsilon}^\infty e^{-y \cosh t + t^{(1+\epsilon)/\epsilon}}\, dt .$$

Since each of these integrals converges,

$$g(\nu) = O\left(e^{\rho^{1+\epsilon}}\right)$$

for large ρ; i.e., the order of $g(\nu)$ is ≤ 1.

Now let $\nu^2 = \lambda$. Then $g(\nu) = g(\lambda^{1/2}) \equiv f(\lambda)$ is an entire function of λ. Moreover, since

$$f(\lambda) = g(\lambda^{1/2}) = O\left(e^{\rho^{1+\epsilon}}\right) = O\left(e^{|\lambda|^{1/2 + \epsilon}}\right)$$

the order of $f(\lambda)$ is $\leq 1/2$. Hence (see [2], pg. 250),

$$f(\lambda) = f(0) \prod_k \left(1 - \frac{\lambda}{\lambda_k}\right),$$

where the λ_k are the zeros of $f(\lambda)$. We note that $f(0) \neq 0$ since $H_0^1(z) \neq 0$ from Corollary 3.3, to be proved later.

In terms of ν, this product becomes

$$H_\nu^1 = e^{-\nu\pi i/2} H_0^1 \prod_k \left(1 - \frac{\nu^2}{\nu_k^2}\right)$$

where the ν_k are the zeros of $H_\nu^1(z)$ for $y > 0$. Moreover

$$f(\lambda) = f(0) \prod_k \left(1 - \frac{\lambda}{\lambda_k}\right) = f(0) \left\{1 - \lambda \sum_k \frac{1}{\lambda_k} + \ldots\right\}$$

$$= f(0) + \lambda f'(0) + \ldots$$

and we find

$$\sum_k \frac{1}{\lambda_k} = -f'(0)/f(0),$$

or

$$\sum_k \frac{1}{\nu_k^2} = -\frac{g'(\nu)}{g(\nu)} \frac{d\nu}{d\lambda}\bigg|_{\nu = 0} = -\frac{\int_0^\infty t^2 e^{iz \cosh t}\, dt}{2\int_0^\infty e^{iz \cosh t}\, dt} .$$

Since $f(\lambda)$ is an entire function of order < 1 and is not a polynomial, it, together with the Hankel function, assumes each value infinitely often. In particular, when $y > 0$, $H_\nu^1(z)$ has an infinite number of zeros (see [2], ch. 8).

We can extend this result to the real axis by applying the same technique to the function $h(\nu)$ defined below. When $x > 0$, from Nicholson's formula ([1], p. 31),

$$(2.1) \qquad h(\nu) \equiv \frac{\pi^2}{8} H_\nu^1(z)\, H_\nu^2(z) = \int_0^\infty K_0(2z \sinh t) \cosh 2\nu t \, dt,$$

where K_0 is the modified Hankel function of order zero. Then, if we put $|\nu| = \rho$,

$$|h(\nu)| \leq \int_0^\infty |K_0(2z \sinh t)|\, e^{2\rho t} dt.$$

Now for any ν and z for which $x > 0$ ([3], p. 181),

$$(2.2) \qquad K_\nu(z) = \int_0^\infty e^{-z \cosh S} \cosh \nu S \, dS.$$

Therefore,

$$|K_0(2z \sinh t)| \leq \int_0^\infty \left| e^{-2z \sinh t \cosh S} \right| dS$$

$$= \int_0^\infty e^{-2x \sinh t \cosh S} dS.$$

Since $\cosh S \geq 1 + \frac{S^2}{2}$, we have for any $a > 0$

$$\int_0^\infty e^{-2a \cosh S} dS \leq e^{-2a} \int_0^\infty e^{-aS^2} dS = \frac{\sqrt{\pi}}{2} \frac{e^{-2a}}{\sqrt{a}};$$

and $$|K_0(2z \sinh t)| \leq \frac{\sqrt{\pi}}{2} \frac{e^{-2x \sinh t}}{\sqrt{x \sinh t}}$$

Thus:

$$|h(\nu)| \leq \frac{\sqrt{\pi}}{2} \int_0^\infty \frac{e^{-2x \sinh t + 2\rho t}}{\sqrt{x \sinh t}} dt.$$

To show that the order of $h(\nu)$ is ≤ 1, it suffices to show that

$$I' = \int_0^\infty \frac{\exp\left[-2x \sinh t + \rho(2t - \rho^\epsilon)\right]}{\sqrt{x \sinh t}} dt < M.$$

Now

$$I' \leq \int_0^{\rho^{\epsilon}/2} \frac{e^{-2x \sinh t}}{\sqrt{x \sinh t}}\,dt + \int_{\rho^{\epsilon}/2}^{\infty} \frac{\exp\left[-2x \sinh t + (2t)^{1+\frac{1}{\epsilon}}\right]}{\sqrt{x \sinh t}}\,dt.$$

Since each integral converges, $h(\nu) = O\left(e^{\rho^{1+\epsilon}}\right)$ for all positive ϵ, and the order of $h(\nu)$ is ≤ 1.

As before, we have $h(\nu) = h(\lambda^{1/2}) \equiv \mathbf{P}(\lambda)$. Since $\cosh 2\nu t$ is an even function of ν, $\mathbf{P}(\lambda)$ is an entire function, and is of order $\leq \frac{1}{2}$. Then there are an infinite number of zeros of $\mathbf{P}(\lambda)$, and hence of H_ν^1 H_ν^2. If $z(=x)$ is real and positive, then

$$\overline{H_\nu^1(x)} = H_{\bar{\nu}}^2(x) \text{ and } H_\nu^1(e^{i\pi}x) = \overline{-H_{\bar{\nu}}^1(x)},$$

and with our previous results we have

Theorem 2.1 If $0 \leq \arg z \leq \pi$, there are an infinite number of zeros of $H_\nu^1(z)$.

Since $\mathbf{P}(\lambda)$ is of order $\leq \frac{1}{2}$,

$$\mathbf{P}(\lambda) = \mathbf{P}(0) \prod_k \left(1 - \frac{\lambda}{\lambda_k}\right),$$

or

$$H_\nu^1(z)\, H_\nu^2(z) = H_0^1(z)\, H_0^2(z) \prod_k \left(1 - \frac{\nu^2}{\nu_k^2}\right)$$

for $x > 0$, where the ν_k are now the zeros of H_ν^1 H_ν^2. Moreover, since

$$\begin{aligned} h(\nu) &= h(0) \prod_k \left(1 - \frac{\nu^2}{\nu_k^2}\right) \\ &= h(0) \left(1 - \nu^2 \sum_k \frac{1}{\nu_k^2} + \cdots\right) \\ &= h(0) + \frac{\nu^2}{2} h''(0) + \cdots, \end{aligned}$$

we have

$$\sum_k \frac{1}{\nu_k^2} = -\frac{h''(0)}{2h(0)} = -\frac{2\int_0^\infty t^2 K_0(2z \sinh t)\, dt}{\int_0^\infty K_0(2z \sinh t)\, dt}.$$

For z real and positive, if $\nu^{(j)}$ is a zero of H_ν^j, we have $\nu_k^{(1)} = \overline{\nu_k^{(2)}}$.

Hence

$$\sum_k \frac{1}{\nu_k^2} = \sum_k \left[\frac{1}{\nu_k^{(1)2}} + \frac{1}{\nu_k^{(2)2}}\right] = \sum_k \left[\frac{1}{\nu_k^{(1)2}} + \overline{\frac{1}{\nu_k^{(1)2}}}\right]$$

and

Theorem 2.2

$$\mathrm{Re} \sum_k \frac{1}{\nu_k^{(1)2}} = -\frac{\int_0^\infty t^2 K_0(2x \sinh t)\, dt}{\int_0^\infty K_0(2x \sinh t)\, dt}.$$

To show that H_ν^1 has only simple zeros, we require the following well-known

Lemma 2.1 Let F(t) be positive, continuous, and decreasing for $t > 0$. Then $\int_0^\infty F(t) \sin t\, dt > 0$.

Proof:

$$\int_0^\infty F(t) \sin t\, dt = \sum_{n=0}^\infty \left[\int_{2n\pi}^{(2n+1)\pi} F(t) \sin t\, dt + \int_{(2n+1)\pi}^{(2n+2)\pi} F(t) \sin t\, dt\right].$$

By the first mean value theorem,

$$\int_{2n\pi}^{(2n+1)\pi} F(t) \sin t\, dt + \int_{(2n+1)\pi}^{(2n+2)\pi} F(t) \sin t\, dt = 2F(t') - 2F(t''),$$

for some t', t'' such that $2n\pi < t' < (2n+1)\pi$ and $(2n+1)\pi < t'' < (2n+2)\pi$. Since F(t) is strictly decreasing, $F(t') - F(t'') > 0$.

We generalize this result in

Lemma 2.2. Let $q(0) = 0$, $q'(t) > 0$, $q''(t) \geq 0$, $p(t) > 0$, $p'(t) < 0$, with $p'(t)$ continuous, for $t > 0$.

Then

$$\int_0^\infty p(t) \sin q(t)\, dt > 0.$$

Proof:

Let $x = q(t)$. Since $q'(t) > 0$, $t = q^{-1}(x) = t(x)$.

Then

$$\int_0^\infty p(t) \sin q(t)\, dt = \int_0^\infty \frac{p[t(x)]}{q'[t(x)]} \sin x\, dx.$$

Since

$$F(x) = \frac{p}{q'} > 0$$

and

$$F'(x) = \frac{p'q' - pq''}{q'^3} < 0,$$

the lemma is a consequence of Lemma 2.1.

Now we can prove the simplicity of the zeros. For if we let $H_\nu^{1}(z) = 0$ where $x > 0$, from a formula of Watson (see [1], p. 30)

$$H_\nu^{(2)}(z) \frac{\partial H_\nu^{(1)}(z)}{\partial \nu} = -\frac{8i}{\pi} \int_0^\infty K_0 (2z \sinh t)\, e^{-2\nu t}\, dt.$$

Since

$$H_\nu^{(2)}(z) \neq 0 \text{ and } K_0 (2z \sinh t) = \int_0^\infty e^{-2 z \sinh t \cosh s}\, ds,$$

it suffices to show that

$$\operatorname{Im} \int_0^\infty \int_0^\infty \exp(-2z \sinh t \cosh s - 2\nu t)\, ds\, dt \neq 0,$$

or

$$\int_0^\infty \int_0^\infty \exp(-2x \sinh t \cosh s - 2\alpha t) \sin(2y \sinh t \cosh s + 2\beta t)\, dt\, ds \neq 0,$$

where we changed the order of integration. As mentioned earlier, we can assume $\alpha \geq 0$ with no loss in generality. Since for $y \geq 0$ we can prove that $\beta > 0$

(see Corollary 3.3 and Theorem 4.1), the integrand, as a function of t, satisfies the requirements of Lemma 2.2 and is therefore positive for each s. Consequently $\frac{\partial H_\nu^1(z)}{\partial \nu} \neq 0$ and we have

Theorem 2.3. The zeros of $H_\nu^1(z)$ are simple when $x > 0$ and $y \geq 0$.

3. Zeros of $H_\nu^1(z)$ for a fixed, complex z.

In this section we shall derive several inequalities relating the zeros ν and the complex arguments z. For the most part, z will be restricted to the upper half plane. If we include the positive real axis this corresponds to the actual physical situation, since the wave number, which is essentially z, lies in the first quadrant. If z is in the lower half plane, the zeros behave entirely differently. This is not too surprising, since the asymptotic expansions or the integral formulas for H_ν^1 reveal a basic difference in the two half planes. For instance, when $y < 0$ and $x \neq 0$ it has been shown that the total number of z's which satisfy $H_\nu^1(z) = 0$ for real ν "is the even integer nearest to $\nu - \frac{1}{2}$, unless $\nu - \frac{1}{2}$ is an integer, in which case the number is $\nu - \frac{1}{2}$ " ([3], pp. 511-2). But for $y > 0$, there are no real values of ν for which $H_\nu^{(1)}(z) = 0$, as is asserted in Corollary 3.3. The following result is due to B. Friedman:

Theorem 3.1. The zeros of $H_\nu^1(iy)$ for $y > 0$ are pure imaginary.

Proof: Consider the differential equation

$$u''(y) - \left(1 + \frac{\nu^2 - \frac{1}{4}}{y^2}\right) u(y) = 0 \quad \text{for } y \geq y_o > 0$$

with $u(y_o) = u(\infty) = 0$. This equation is of the Sturm-Liouville type. It has the solution $u(y) = \sqrt{y}\, H_\nu^1(iy)$. Since the eigenvalues ν^2 are real, ν is

either real or pure imaginary. The former is impossible, since by Heine's formula

$$H_\nu^1(iy_o) = -\frac{2i}{\pi} e^{-\nu\pi i/2} \int_o^\infty e^{-y_o \cosh t} \cosh \nu t \, dt \neq 0$$

for ν real. Then the zeros of $H_\nu^1(iy)$ are pure imaginary.

We have something of a converse in

Theorem 3.2. Given $\beta \neq 0$, there exists a positive y such that $H_{i\beta}^1(iy) = 0$.

Proof: For b real, we have (see [1], p. 34)

$$\int_o^\infty e^{-t \cosh b} H_{i\beta}^{(1)}(it) \, dt = -\frac{2i \, e^{\beta\pi/2} \sin(\beta b)}{\sinh \beta\pi \sinh b} .$$

Choose $\beta = \frac{\pi}{b}$. Then the integral vanishes. Now $H_{i\beta}^1(it)$ is pure imaginary, since

$$-H_{i\beta}^1(it) = H_{i\beta}^2(-it) = \overline{H_{i\beta}^1(it)}.$$

Because of $\exp(-t \cosh b) > 0$, $H_{i\beta}^1(it)$ must vanish for some $t > 0$. Since b, and hence β, was arbitrary, the theorem is proved.

In this case we can show that $|\beta| > y$ by means of the following classical argument. The function $H_\nu^1(ze^t) = u(t) + iv(t)$ satisfies the equation

$$\frac{d^2}{dt^2} H_\nu^1(ze^t) + (z^2 e^{2t} - \nu^2) H_\nu^1(ze^t) = 0.$$

In terms of its real and imaginary parts, this becomes

$$u'' + \left[(x^2 - y^2) e^{2t} - (\alpha^2 - \beta^2)\right] u - 2(xye^{2t} - \alpha\beta) v = 0$$

$$v'' + \left[(x^2 - y^2) e^{2t} - (\alpha^2 - \beta^2)\right] v + 2(xye^{2t} - \alpha\beta) u = 0.$$

First we multiply these by v and u respectively and subtract, getting

$$u''v - u v'' - 2(xye^{2t} - \alpha\beta)(u^2 + v^2) = 0.$$

If we multiply instead by u and v respectively and add, we obtain

$$uu'' + vv'' + \left[(x^2 - y^2) e^{2t} - (\alpha^2 - \beta^2)\right](u^2 + v^2) = 0.$$

Integrating these equations then gives us

Theorem 3.3. If $H_\nu^1(z\, e^{t_o}) = 0$ and $y > 0$, then

$$(3.1)\quad \int_{t_o}^{\infty} (xye^{2t} - \alpha\beta)\,(u^2 + v^2)\,dt = 0$$

and

$$(3.2)\quad \int_{t_o}^{\infty} \left[(x^2 - y^2)\, e^{2t} - (\alpha^2 - \beta^2)\right]\,(u^2 + v^2)\,dt = \int_{t_o}^{\infty} (u'^2 + v'^2)\,dt > 0.$$

We now let $z = \nu$. Then equation (3.1) becomes

$$\alpha\beta \int_{t_o}^{\infty} (e^{2t} - 1)\,(u^2 + v^2)\,dt = 0.$$

If $t_o \geq 0$, then we must have $\alpha\beta = 0$. But then $\alpha = 0$ since $\beta = y > 0$, and (3.2) becomes

$$\beta^2 \int_{t_o}^{\infty} (1 - e^{2t})\,(u^2 + v^2)\,dt > 0.$$

This, however, is impossible if $t_o \geq 0$. Therefore, t_o must be negative.

We have thus obtained the following result.

Corollary 3.1. If $H_\nu^1(a\nu) = 0$ for $a > 0$ and $\beta > 0$, then $a < 1$.

We have as a special case

Corollary 3.2. If $H_{i\beta}^1(iy) = 0$ for $y > 0$, then $|\beta| > y$.

We can exploit equations (3.1) and (3.2) somewhat further. For instance let $x > 0$, and recall that $y > 0$. If t is sufficiently large, $x\,y\,e^{2t} - \alpha\beta > 0$. Then for the integral in (3.1) to vanish, since the above function is monotonic in t, we must have $x\,y\,e^{2t_o} - \alpha\beta < 0$. Similarly, if $x < 0$, the inequality is reversed. Finally if $x = 0$, from Theorem 3.1, $\alpha = 0$. This establishes

Corollary 3.3. If $H_\nu^1(z) = 0$ for $y > 0$, then $\operatorname{sgn}(\alpha\beta - xy) = \operatorname{sgn} x$.

Moreover, if $y = 0$, then $H_\alpha^1(x) = \overline{H_\alpha^2(x)}$. But these, as linearly independent solutions of Bessel's equation, cannot vanish simultaneously. Together with Corollary 3.3, this proves that if $y \geq 0$, then $H_\alpha^1(z) \neq 0$, a result first obtained by Macdonald ([3], p. 511).

Again from equation (3.1), if $x\,\alpha\,\beta = 0$, then $x = \alpha\,\beta = 0$. By considering each quantity separately, we obtain the following results. First we let $\alpha = 0$; then we have

Corollary 3.4 If $H^1_{i\beta}(z) = 0$ and $y > 0$, then $z = iy$.

Secondly, letting $\beta = 0$, we obtain MacDonald's result. Finally, if $x = 0$, we find $H^1_\nu(iy) = 0$ implies $\nu = \alpha$ or $\nu = i\beta$. But $\nu = \alpha$ is impossible, by Heine's formula, and we again obtain Theorem 3.1.

If we now consider equation (3.2) and assume $x^2 - y^2 \leq 0$, the same reasoning as before indicates that $(x^2 - y^2)e^{2t_o} - (\alpha^2 - \beta^2) > 0$. This is the content of

Corollary 3.5. If $H^1_\nu(z) = 0$ with $x^2 \leq y^2$ and $y > 0$, then $\alpha^2 - \beta^2 < x^2 - y^2$.

Further results restricting the location of the zeros can be obtained from Hankel's asymptotic formula. This expression is valid not only for the magnitude of r of z large compared to that of ν and 1, when it attains greatest accuracy, but also for small values of z. If we consider only the first term of the expansion, Hankel's formula is

$$(3.3) \qquad H^1_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}(1+R).$$

We shall find conditions under which the remainder R has magnitude less than unity, whence $H^1_\nu(z) \neq 0$. First we shall use Weber's form for the remainder.

The derivation of this formula for real values of ν is reproduced in Watson ([3], pp. 211-2). This can be extended to complex values by only slight changes. Without going through this lengthy analysis, we merely state the following inequality for R. If

$$2\,|z| = 2r \geq \alpha - \tfrac{1}{2} > 0,\ \beta \geq 0, \text{ and } x > 0,$$

then

$$|R| \leq \frac{\pi}{2r}\left|\nu^2 - \tfrac{1}{4}\right| \left(1 - \frac{\alpha - \frac{1}{2}}{2r}\right)^{-2\alpha-1} \left|\frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})}\right|^2 .$$

But it can easily be shown that

$$\left(1 - \frac{\alpha - \frac{1}{2}}{2r}\right)^{-2\alpha - 1} \leq \frac{2r - \alpha + \frac{1}{2}}{2r - (\alpha + 1)(2\alpha - 1)}$$

when $(\alpha + 1)\left(\alpha - \frac{1}{2}\right) < r$, and that

$$\left|\frac{\Gamma(\alpha + \frac{1}{2}i)}{\Gamma(\nu + \frac{1}{2}i)}\right|^2 \leq \frac{1}{\beta\pi} \sinh \beta\pi .$$

Then

$$|R| \leq \frac{\pi}{2r} \left|\nu^2 - \frac{1}{4}\right| \frac{2r - \alpha + \frac{1}{2}}{2r - (\alpha + 1)(2\alpha - 1)} \frac{\sinh \beta\pi}{\beta\pi} ,$$

which is less than unity when

$$0 < 4r^2 - 2r \left[(\alpha + 1)(2\alpha - 1) + \frac{\left|\nu^2 - \frac{1}{4}\right| \sinh \beta\pi}{\beta}\right] + \frac{(\alpha - \frac{1}{2})\left|\nu^2 - \frac{1}{4}\right| \sinh \beta\pi}{\beta} .$$

This is a quadratic in r, giving us a simpler inequality in

Theorem 3.4. If $H_\nu^1(z) = 0$ for $\alpha > \frac{1}{2}$, $\beta \geq 0$, and $x > 0$, then $2|z| \leq (\alpha + 1)(2\alpha - 1) + \left|\nu^2 - \frac{1}{4}\right| \frac{\sinh \beta\pi}{\beta}$.

We can complement this result insofar as α is concerned by considering another form for the remainder due to Schläfli ([3], p. 219). For

$$|\alpha| < \frac{3}{2} \text{ and } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} ,$$

the remainder R of Hankel's formula satisfies

$$(3.4) \qquad |R| \leq \left|\frac{\cos \nu\pi}{\cos \alpha\pi}\right| \frac{\left|\alpha^2 - \frac{1}{4}\right| c}{2r} \leq \frac{\cosh \beta\pi}{|\cos \alpha\pi|} \frac{\left|\alpha^2 - \frac{1}{4}\right| c}{2r}$$

where $c = 1$ when $y \geq 0$ and $c = |\sec \arg z|$ when $y \leq 0$. Then if $H_\nu^1(z) = 0$, we must have $|R| = 1$, which gives the following results.

Theorem 3.5. If $H_\nu^1(z) = 0$ for $|\alpha| < \frac{3}{2}$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, then

$$2r \leq \frac{c \cosh \beta\pi}{|\cos \alpha\pi|} \left| \alpha^2 - \frac{1}{4} \right|$$

where c is defined as above.

Since $|\alpha| < \frac{3}{2}$, we obtain the following less accurate but simpler inequality for $y \geq 0$:

$$r \leq \frac{\cosh \beta\pi}{|\cos \alpha\pi|} .$$

4. Roots of $H_\nu^{(1)}(z) = 0$ for real, positive values of z

Most of the results of the preceding section are valid only when $y > 0$. In this section we shall restrict z to the positive real axis.

The first result will follow from a lemma which will be useful later on.

Lemma 4.1. If $H_\nu^1(x) = 0$ and $x > 0$, then

$$\int_x^\infty |H_\nu^1(t)|^2 \frac{dt}{t} = \frac{e^{\beta\pi}}{\alpha\beta\pi} .$$

Proof: We consider the self adjoint form of Bessel's differential equation for $H_\nu^1(x)$ and $H_\nu^2(x)$. Going through the already familiar steps of multiplying each equation by the other function, subtracting, and integrating, we obtain

$$\int_x^\infty \frac{\nu^2 - \overline{\nu}^2}{t} H_\nu^1(t) H_\nu^2(t) \, dt = t \left[H_\nu^1(t) \frac{d}{dt} H_\nu^2(t) - H_\nu^2(t) \frac{d}{dt} H_\nu^1(t) \right]_x^\infty$$

or

$$4i\alpha\beta \int_x^\infty |H_\nu^1(t)|^2 \frac{dt}{t} = \frac{4i}{\pi} e^{\beta\pi}$$

from the behavior of the Hankel functions at infinity, and the fact that $H_\nu^2(t) = \overline{H_\nu^1(t)}$.

Since the integral in the last equation converges, we have

Theorem 4.1. If $H_\nu^1(x) = 0$ for $x > 0$, then $\alpha\,\beta > 0$.

As far as α is concerned, this tells us only that $|\alpha| > 0$. We can improve this inequality, but first we require the following well-known extension of Lemma 2.1.

Lemma 4.2. Let $\int_0^\infty f(t) \cos bt\, dt$ exist for $b \neq 0$, where $f(t)$ is twice differentiable for $t > 0$ with $f'(t) < 0$ and $f''(t) > 0$. Then $\int_0^\infty f(t) \cos bt\, dt > 0$.

Proof: Integrating by parts, we obtain

$$\int_0^\infty f(t) \cos bt\, dt = \frac{1}{b} f(t) \sin bt \Big|_0^\infty - \frac{1}{b} \int_0^\infty f'(t) \sin bt\, dt.$$

Since $\int_0^\infty f(t) \cos bt\, dt$ exists with $f'(t) < 0$, we must have $\lim_{t \to \infty} f(t) = 0$. And since $\int_0^\infty f(t) \cos bt\, dt$ exists and $f(t)$ is monotonic, $f(t) = o(t^{-1})$ as $t \to 0$; hence $\lim_{t \to 0} f(t) \sin bt = 0$. Then

$$\int_0^\infty f(t) \cos bt\, dt = -\frac{1}{b} \int_0^\infty f'(t) \sin bt\, dt,$$

and an application of Lemma 2.1 completes the proof.

We are now able to prove

Theorem 4.2. Let $H_\nu^1(x) = 0$ with $x > 0$. Then $|\alpha| > x - \sqrt{x/2}$.

Proof: From Nicholson's formula (2.1) for $x > 0$,

$$\operatorname{Re}\left[\frac{\pi^2}{8} H_\nu^1(x)\, H_\nu^2(x)\right] = \int_0^\infty K_0(2x \sinh t) \cosh 2\alpha t \cos 2\beta t\, dt.$$

We shall apply the preceding lemma to find conditions under which this does not vanish. We set

$$f(t) = K_0(2x \sinh t) \cosh 2\alpha t$$

$$= \int_0^\infty e^{-2x \sinh t \cosh s} \cosh 2\alpha t\, ds$$

for $t > 0$. Then

$$f'(t) = \int_0^\infty e^{-2x \sinh t \cosh s} \left[-2x \cosh s \cosh t \cosh 2\alpha t + 2\alpha \sinh 2\alpha t\right] ds < 0$$

when $|\alpha| \leq x$, and

$$f''(t) = \int_0^\infty e^{-2x \sinh t \cosh s} A(\alpha,x,s,t)\, ds$$

where

$$A = 4x^2 \cosh^2 s \cosh^2 t \cosh 2\alpha t - 8\alpha x \cosh s \cosh t \sinh 2\alpha t - 2x \cosh s \sinh t \cosh 2\alpha t + 4\alpha^2 \cosh 2\alpha t.$$

Let $\alpha \geq 0$. Then

$$A \geq \left[4x^2 \cosh^2 s \cosh^2 t - 2x(4\alpha+1) \cosh s \cosh t + 4\alpha^2\right] \cosh 2\alpha t$$

which is non-negative for all s and t when

$$x \geq \frac{4\alpha + 1 + \sqrt{8\alpha + 1}}{4} .$$

Thus f(t) satisfies the conditions of Lemma 4.2 and we find that if $H_\nu^1(x) = 0$ with $\alpha \geq 0$, then

$$x < \frac{4\alpha + 1 + \sqrt{8\alpha + 1}}{4} .$$

But this can be shown to imply $\alpha > x - \sqrt{x/2}$.

If we restrict $|\alpha|$ to less than $\frac{1}{2}$, we obtain another inequality.

Theorem 4.3. Let $H_\nu^1(x) = 0$ with $x > 0$ and $|\alpha| < \frac{1}{2}$.

Then

$$x \leq \beta \tan \alpha\pi + \frac{1}{2} .$$

Proof: Using a formula due to Watson ([3], p. 445), we find for $x > 0$ and $|\alpha| < \frac{1}{2}$ that

$$\frac{8i \sin 2\alpha\pi}{\pi^2} \int_0^\infty K_{2\alpha}(2x \sinh t)\, e^{-2i\beta t}\, dt$$

$$= H_{-\bar\nu}^1(x)\, H_\nu^2(x) - H_\nu^1(x)\, H_{-\bar\nu}^2(x)$$

$$= |H_\nu^2(x)|^2 e^{i\bar\nu\pi} - |H_\nu^1(x)|^2 e^{-i\bar\nu\pi} .$$

Then

$$(4.1)\quad \cos \alpha\pi \int_0^\infty K_{2\alpha}(2x \sinh t) \cos 2\beta t \, dt = \frac{\pi^2}{16} \left(|H_\nu^2|^2 e^{\beta\pi} + |H_\nu^1|^2 e^{-\beta\pi} \right),$$

$$(4.2)\quad \sin \alpha\pi \int_0^\infty K_{2\alpha}(2x \sinh t) \sin 2\beta t \, dt = \frac{\pi^2}{16} \left(|H_\nu^2|^2 e^{\beta\pi} - |H_\nu^1|^2 e^{-\beta\pi} \right) .$$

After subtracting we obtain

$$\frac{\pi^2}{8} |H_\nu^1(x)|^2 e^{-\beta\pi} = \int_0^\infty K_{2\alpha} (2x \sinh t) \cos (2\beta t + \alpha\pi) \, dt.$$

Dividing both sides by $\cos \alpha\pi$ and integrating by parts give us

$$\frac{\pi^2 e^{-\beta\pi} |H_\nu^1|^2}{8 \cos \alpha\pi} = - \int_0^\infty \sin 2\beta t \left\{ \frac{1}{2\beta} \frac{d}{dt} K_{2\alpha}(2x \sinh t) + \tan \alpha\pi \, K_{2\alpha}(2x \sinh t) \right\} dt.$$

Applying equation (2.2) for $K_\nu(z)$ yields

$$\frac{\pi^2 e^{-\beta\pi} |H_\nu^1|^2}{8 \cos \alpha\pi} = \int_0^\infty f(t) \sin 2\beta t \, dt$$

where

$$f(t) \equiv \int_0^\infty e^{-2x \sinh t \cosh s} \cosh 2\alpha s \left(\frac{x}{\beta} \cosh t \cosh s - \tan \alpha\pi \right) ds.$$

Then

$$f'(t) = \int_0^\infty e^{-2x \sinh t \cosh s} \cosh 2\alpha s \cosh s \cdot \left[\frac{x}{\beta} \sinh t \cosh s - 2x \cosh t \left(\frac{x}{\beta} \cosh t \cosh s - \tan \alpha\pi \right) \right] ds.$$

If $\beta > 0$ and $x > \beta \tan \alpha\pi$, then $f(t) > 0$; also $f'(t) < 0$ when $\sinh t < 2 \cosh t \, (x \cosh t \cosh s - \beta \tan \alpha\pi)$. Simplifying this and applying Lemma 2.1 complete the proof.

We can close the α interval under consideration in the following manner. If we let $\nu = \frac{1}{2} + i\beta$, then

$$\frac{4}{\pi t} = i(H_\nu^1 H_\nu^2 - 1 - H_{\nu-1}^1 H_\nu^2) = i(H_\nu^1 H_{-\nu}^2 - H_{-\nu}^1 H_\nu^2)$$

$$= |H_\nu^1|^2 e^{-\beta\pi} + |H_\nu^2|^2 e^{\beta\pi},$$

where the argument of both Hankel functions is t. This is easily proved for $\alpha = -\frac{1}{2}$, and incidentally provides an extension of (4.1). Now if we divide both sides by t and integrate from x to ∞, applying Lemma 4.1 we find

$$\frac{4}{\pi x} = \frac{2}{\beta\pi} + e^{\beta\pi} \int_x^\infty |H_\nu^2(t)|^2 \frac{dt}{t} ,$$

whence

Theorem 4.4. Let $H_\nu^1(x) = 0$ with $x > 0$, $\alpha = \pm \frac{1}{2}$.
Then $|\beta| > \frac{x}{2}$.

If we apply Lemma 2.1 to equation (4.2), we obtain

Theorem 4.5. If $x > 0$ and $|\alpha| < \frac{1}{2}$, then

$$\operatorname{sgn}\left[|H_\nu^2(x)|\, e^{\beta\pi} - |H_\nu^1(x)|\right] = \operatorname{sgn} \alpha\beta.$$

5. Behavior of the zeros as $x \to 0$

The inequalities $|\alpha| > x - \sqrt{x/2}$ and $x \leq \beta \tan \alpha\pi + \frac{1}{2}$, which we found in the preceding section, give us no information when $0 < x < \frac{1}{2}$. We can partially explore this range by letting x tend to zero, which enables us to employ approximation formulas of a relatively simple nature.

Let us consider Lemma 4.1:

$$\int_x^\infty |H_\nu^1(t)|^2 \frac{dt}{t} = \frac{e^{\beta\pi}}{\alpha\beta\pi}$$

where $H_\nu^1(x) = 0$ for $x > 0$. If we let $x \to 0$, the integral diverges, whence either $\alpha\beta \to 0$ or $\beta \to \infty$. The latter possibility is eliminated since $H_\nu^1 = 0$ implies $H_{-\nu}^1 = 0$; and if $\beta \to \infty$ then $\frac{e^{-\beta\pi}}{\alpha\beta\pi} \to 0$ unless $\alpha \to 0$, whereas the integral diverges. Consequently either $\alpha \to 0$ or $\beta \to 0$.

We can show that at least α approaches zero.

Theorem 5.1. Let $H_\nu^1(x) = 0$. If $x \longrightarrow 0+$, then $\alpha \longrightarrow 0$.

Proof: Let $H_\nu^1(t) \equiv u(t) + iv(t)$. Then separating the self-adjoint form of Bessel's equation into real and imaginary parts, we obtain

$$(tu')' + (t - \frac{\alpha^2 - \beta^2}{t})\, u = -\frac{2\alpha\beta}{t}\, v$$

and

$$(tv')' + (t - \frac{\alpha^2 - \beta^2}{t})\, v = \frac{2\alpha\beta}{t}\, u.$$

Multiplying by u and v respectively and adding:

$$(5.1) \qquad (tu')'\, u + (tv')'\, v + (t - \frac{\alpha^2 - \beta^2}{t})\,(u^2 + v^2) = 0.$$

Integrating from x to ∞

$$\int_x^\infty \left[-t\,(u'^2 + v'^2) + (t - \frac{\alpha^2 - \beta^2}{t})\,(u^2 + v^2) \right] dt = 0.$$

From Lemma 4.1, this simplifies to

$$\int_x^\infty t \left[|H_\nu^1|^2 - \left| \frac{dH_\nu^1}{dt} \right|^2 \right] dt = \frac{(\alpha^2 - \beta^2)\, e^{\beta\pi}}{\alpha\beta\pi}\,.$$

Let $x \longrightarrow 0$. If $\alpha > \epsilon$ for some $\epsilon > 0$, then as shown below the integral diverges to $-\infty$, and $\alpha^2 < \beta^2$. But since $\alpha \longrightarrow 0$ or $\beta \longrightarrow 0$, this implies that $\alpha \longrightarrow 0$.

To show that the integral diverges as stated we consider the integrand for small t, and $\alpha > \epsilon > 0$. Then

$$|H_\nu^1(t)| = \left| \frac{J_{-\nu}(t) - e^{-i\nu\pi}\, J_\nu(t)}{\sin \nu\pi} \right| \sim \frac{1}{\pi} \left(\frac{t}{2}\right)^{-\alpha} \left| \Gamma(\nu) \right|.$$

Similarly

$$\left| \frac{d}{dt}\, H_\nu^1(t) \right| \sim \frac{1}{\pi} \left(\frac{t}{2}\right)^{-\alpha-1} \left| \Gamma(\nu + 1) \right|.$$

Thus the second term dominates the integrand, and the integral diverges to $-\infty$. This completes the proof.

For small x, we now find a relation involving both α and β.

Theorem 5.2. Let $H_\nu^1(x) = 0$. If $x \to 0+$, then

$$\lim\left(\alpha \log \frac{x}{2} + \frac{\beta\pi}{2}\right) = 0.$$

Proof:

$$0 = H_\nu^1(x) = \frac{J_{-\nu}(x) - e^{-i\nu\pi} J_\nu(x)}{i \sin \nu\pi} \sim \frac{1}{\pi}\left[\left(\frac{x}{2}\right)^{-\nu} \Gamma(\nu) + e^{-i\nu\pi}\left(\frac{x}{2}\right)^{\nu} \Gamma(-\nu)\right].$$

Let $x \to 0+$, and restrict α to positive values also. Then

$$\lim_{x \to 0+} \left|\left(\frac{x}{2}\right)^{-\nu} \Gamma(\nu)\right| = \lim e^{-\alpha \log x/2} |\Gamma(\nu)| = c\,|\Gamma(\nu_o)|$$

where

$$c \geq 1 \text{ and } \nu_o = i\beta_o = \lim \nu.$$

Also

$$\lim \left|\left(\frac{x}{2}\right)^{\nu} \Gamma(-\nu) e^{-i\nu\pi}\right| = \lim e^{\alpha \log x/2 + \beta\pi} |\Gamma(-\nu)| = \frac{1}{c} e^{\beta_o \pi} |\Gamma(-\nu_o)|.$$

Since $\nu_o = i\beta_o$, $\left|\Gamma(\nu_o)\right| = \left|\Gamma(-\nu_o)\right|$; and since $H_\nu^1(x) = 0$ in the limit also, the two limits above must be equal, whence $c = e^{\beta_o \pi/2} = \lim e^{-\alpha \log x/2}$.

6. The derivative of ν

We can obtain some interesting results not only for small x, as in the last section, but for any x by examining $\frac{d\nu}{dx}$. However, it is not much more difficult to consider complex values of the argument. We shall derive a formula which was first obtained by Watson (see [3], p. 508),

$$\frac{dc}{d\nu} = 2c \int_o^\infty K_o(2c \sinh t)\, e^{-2\nu t}\, dt,$$

where ν is real and c is a positive zero (in z) of the cylindrical function $J_\nu(z) \cos s + Y_\nu(z) \sin s$ for constant s. This form immediately excludes H_ν^1. We shall generalize it to apply to complex values of z and ν, as well as to any function of the form $C_\nu(z) = c_1 J_\nu(z) + c_2 Y_\nu(z)$, where c_1 and c_2 are independent of both z and ν.

All the results of this section will be corollaries of

Theorem 6.1. If $C_\nu(z)$ is of the above form, with $x > 0$, then any zero ν satisfies

$$(6.1) \qquad 2z \frac{d\nu}{dz} \int_o^\infty K_o(2z \sinh t)\, e^{-2\nu t}\, dt = 1.$$

Proof: Let $B_\nu(z) = b_1 J_\nu(z) + b_2 Y_\nu(z)$ be a function of the same type as $C_\nu(z)$ and linearly independent of it. Then

$$\delta = b_1 c_2 - b_2 c_1 \neq 0.$$

The Wronskians of these functions with respect to ν and z are respectively

$$B_\nu \frac{\partial C_\nu}{\partial \nu} - C_\nu \frac{\partial B_\nu}{\partial \nu} = \delta \left(J_\nu \frac{\partial Y_\nu}{\partial \nu} - Y_\nu \frac{\partial J_\nu}{\partial \nu} \right)$$

and

$$B_\nu \frac{dC_\nu}{dz} - C_\nu \frac{dB_\nu}{dz} = \frac{2\delta}{\pi z} .$$

If $C_\nu(z) \not\equiv 0$, then

$$B_\nu \frac{dC_\nu}{dz} + B_\nu \frac{d\nu}{dz} \frac{\partial C_\nu}{\partial \nu} = 0,$$

whence $\frac{d\nu}{dz} \neq 0$. Each term can now be replaced by one of the above Wronskians, giving us

$$\frac{2\delta}{\pi z} + \delta \left(J_\nu \frac{\partial Y_\nu}{\partial \nu} - Y_\nu \frac{\partial J_\nu}{\partial \nu} \right) \frac{d\nu}{dz} = 0.$$

The quantity in parentheses, however, is a constant multiple of the integral appearing below ([1], p. 30); hence

$$\frac{2}{\pi z} - \frac{4}{\pi} \frac{d\nu}{dz} \int_0^\infty K_0(2z \sinh t)\, e^{-2\nu t} dt = 0.$$

We shall restrict C_ν to H_ν^1, although many of the following results are valid for the general cylindrical function.

Corollary 6.1. Let $H_\nu^1(x) = 0$ for α and x positive. Then $\alpha'(x) > 0$ when $\alpha \geq \frac{1}{8}$ or $x \geq \frac{1}{4}$; and $\beta'(x) > 0$.

Proof: Solving (6.1) for $\frac{d\nu}{dx}$ and separating real and imaginary parts, we find

$$\frac{d\alpha}{dx} = 2x \left| \frac{d\nu}{dx} \right|^2 \int_0^\infty K_0 (2x \sinh t)\, e^{-2\alpha t} \cos 2\beta t \, dt$$

and

$$\frac{d\beta}{dx} = 2x\left|\frac{dv}{dx}\right|^2 \int_0^\infty K_0\,(2x \sinh t)\, e^{-2\alpha t} \sin 2\beta t \, dt.$$

Consider the kernel

$$f(t) \equiv K_0(2x \sinh t)\, e^{-2\alpha t}$$

$$= \int_0^\infty e^{-2x \sinh t \cosh s - 2\alpha t}\, ds;$$

$$f'(t) = \int_0^\infty e^{-2x \sinh t \cosh s - 2\alpha t}(-2x \cosh t \cosh s - 2\alpha)ds;$$

$$f''(t) = \int_0^\infty e^{-2x \sinh t \cosh s - 2\alpha t} g(s)ds$$

where

$$g(s) = (2x \cosh t \cosh s + 2\alpha)^2 - 2x \sinh t \cosh s.$$

Since $f(t) > 0$ and $f'(t) < 0$, the integral expression for β' indicates that $\beta'(x) > 0$. If we can prove $f''(t) > 0$, it would follow from Lemma 4.2 that $\alpha' > 0$. It suffices to consider $g(s)$.

$$g(s) = (2x \cosh s)^2 \cosh^2 t + (2x \cosh s)(4\alpha \cosh t - \sinh t) + 4\alpha^2 \geq 0,$$

provided the discriminant of this quadratic in $2x \cosh s$ is non-positive; i.e.,

$$-8\alpha \sinh t \cosh t + \sinh^2 t \leq 0,$$

or, a fortiori, $\alpha \geq \frac{1}{8}$. We can show $f''(t) > 0$ also when $x \geq \frac{1}{4}$; for

$$g(s) \geq 2x \cosh s\,(2x \cosh^2 t - \sinh t)$$

$$= 2x \cosh s\,(2x \sinh^2 t - \sinh t + 2x) \geq 0$$

when $1 - 16x^2 \leq 0$. Then f satisfies the conditions of Lemma 4.2 for the above values of α or x.

Moreover, with $\alpha > 0$, we have from (6.1)

$$1 \leq 2x\,|\nu'| \int_0^\infty K_0(2x \sinh t)\,dt$$

$$= \frac{\pi^2}{4}\, x\,|\nu'|\, H_0^1(x)\, H_0^2(x) = \frac{\pi^2}{4}\, x\,|\nu'|\,|H_0^1(x)|^2$$

where the first equation results from (2.1). Since

$$|H_0^1(x)| \leq \left(\frac{2}{\pi x}\right)^{1/2}\left(1 + \frac{1}{8x}\right),$$

as is immediately verified by using Hankel's asymptotic formula (3.3) with Schläfli's form for the remainder (3.4), we obtain

Corollary 6.2. If $H_\nu^1(x) = 0$ with $x > 0$, then

$$|\nu'(x)| \geq \frac{4}{\pi^2 x |H_0^1(x)|^2} \geq \frac{2}{\pi\left(1 + \frac{1}{8x}\right)^2}\,.$$

Since

$$\lim_{x \to 0}\; x\,|H_0^1(x)|^2 = 0\,,$$

we note that

$$\lim_{x \to 0}\; |\nu'(x)| = \infty\,.$$

These corollaries have so far been restricted to real values x. For complex values z, we get the following analogue of Corollary 6.1, with the inequalities suffering slightly and no longer independent.

Corollary 6.3. Let $H_\nu^1(z) = 0$ with x, y, α, β positive. Then

$$x\frac{\partial \alpha}{\partial x} - y\frac{\partial \beta}{\partial x} > 0 \text{ when } x \geq \frac{1}{2} \text{ and } \alpha \geq \frac{\sqrt{2} - 1}{2}\,;$$

and

$$x\frac{\partial \beta}{\partial x} + y\frac{\partial \alpha}{\partial x} > 0.$$

Proof: From (2.2) for $x > 0$

$$\int_0^\infty K_0(2z \sinh t)\, e^{-2\nu t}\,dt = \int_0^\infty \int_0^\infty e^{-2z \sinh t \cosh s - 2\nu t}\,ds\,dt$$

$$= \int_0^\infty \int_0^\infty e^{-p - iq}\,ds\,dt$$

where

$$p(t) = 2x \sinh t \cosh s + 2\alpha t$$

and

$$q(t) = 2y \sinh t \cosh s + 2\beta t.$$

Substituting this in the following form of (6.1),

$$z\nu' = 2|z\nu'|^2 \int_0^\infty \overline{K_0(2z \sinh t)\, e^{-2\nu t}}\, dt,$$

and separating real and imaginary parts, we get

$$x\alpha_x - y\beta_x = 2|z\nu'|^2 \int_0^\infty \int_0^\infty e^{-p} \cos q \, dt \, ds$$

and

$$x\beta_x + y\alpha_x = 2|z\nu'|^2 \int_0^\infty \int_0^\infty e^{-p} \sin q \, dt \, ds$$

where we changed the order of integration.

Let $P(t) = e^{-p}$. Since $P' < 0$, $q' > 0$ and $q'' > 0$,

we apply Lemma 2.2 to prove the second part of the corollary.

To prove the first part, we shall apply Lemma 4.2 to the integrand $\int_0^\infty P(t) \cos q(t)\, dt$, which must first be put into suitable form. We let $w = q(t)$. Since $q'(t) > 0$, $t = q^{-1}(w) = t(w)$. Then

$$\int_0^\infty P(t) \cos q(t)\, dt = \int_0^\infty \frac{P[t(w)]}{q'[t(w)]} \cos w \, dw.$$

Consider the function

$$F(w) = \frac{P[t(w)]}{q'[t(w)]}.$$

Then

$$F'(w) = \frac{P'q' - Pq''}{q'^3} < 0,$$

and

$$q'^5 F''(w) = q'(P''q' - Pq''') + 3q''(Pq'' - P'q').$$

The second term is positive. The first term, apart from the positive factor $q'e^{-p}$, satisfies

$$g(s)\left[2y \cosh t \cosh s + 2\beta\right] - 2y \cosh t \cosh s \geq \left[g(s) - 1\right] 2y \cosh t \cosh s$$

when $g(s)$, defined in the proof of Corollary 6.1, is positive, or $\alpha \geq \frac{1}{8}$ or $x \geq \frac{1}{4}$. But $g(s) - 1 \geq 0$ when $x \geq \frac{1}{2}$ and $8\alpha x + 4\alpha^2 \geq 1$ or $\alpha \geq \frac{\sqrt{2} - 1}{2}$. Consequently $F''(w) > 0$ and an application of Lemma 4.2 completes the proof.

The inequalities occurring in the preceding corollary can be expressed more simply in terms of the partial derivatives with respect to the polar coordinates r and ϕ of the z-plane.

Corollary 6.4. Let $H_\nu^1(z) = 0$ with x, y, α, β positive. Then

$$\frac{\partial \alpha}{\partial \phi} < 0 \text{ and } \frac{\partial \beta}{\partial r} > 0.$$

If in addition $x \geq \frac{1}{2}$ and $\alpha \geq \frac{\sqrt{2} - 1}{2}$, then

$$\frac{\partial \alpha}{\partial r} > 0 \text{ and } \frac{\partial \beta}{\partial \phi} > 0.$$

Proof: With the help of the Cauchy-Riemann equations for the analytic function $\nu(z)$, we find

$$\beta_r = \frac{x\beta_x + y\beta_y}{r} = \frac{x\beta_x + y\alpha_x}{r} > 0,$$

and

$$\alpha_\phi = -y\alpha_x + x\alpha_y = -y\alpha_x - x\beta_x < 0.$$

If x and α are restricted to the above values, then

$$\alpha_r = \frac{x\alpha_x + y\alpha_y}{r} = \frac{x\alpha_x - y\beta_x}{r} > 0$$

and

$$\beta_\phi = -y\beta_x + x\beta_y = -y\beta_x + x\alpha_x > 0.$$

7. Behavior of the large zeros

In this section we shall approximate the large zeros by considering the behavior of $H_\nu^1(z)$ for $|\nu|$ large compared to 1 and $|z|$.

The first Hankel function may be defined by

$$(7.1)\qquad H_\nu^1(z) = \frac{J_{-\nu}(z) - e^{-i\nu\pi}J_\nu(z)}{i \sin \nu\pi}$$

of its limit if the order is an integer. It follows that

$$(7.2)\qquad -i\nu \sin \nu\pi \; H_\nu^1(2z) = A(\nu) + D(\nu)$$

where

$$(7.3)\qquad A(\nu) = \frac{z^{-\nu}}{\Gamma(-\nu)} + \frac{e^{-i\nu\pi} z^\nu}{\Gamma(\nu)}$$

and

$$(7.4)\qquad D(\nu) = \nu \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{k!} \left[- \frac{z^{-\nu}}{\Gamma(-\nu+k+1)} + \frac{e^{-i\nu\pi} z^\nu}{\Gamma(\nu+k+1)} \right] \equiv D_1(\nu) + D_2(\nu).$$

For ν large, $D(\nu)$ becomes negligible, so we shall investigate the zeros of $A(\nu)$. Then we shall show that the large zeros of H_ν^1 approximate those of $A(\nu)$. Throughout, z will be fixed.

Using Stirling's approximation for the gamma function, if $z = re^{i\phi}$ and $\nu = \rho e^{i\theta}$ with $0 < \theta < \pi$, we find for large ν that

$$(7.5)\qquad \sqrt{2\pi}\; A(\nu) \sim \sqrt{2\pi}\; B(\nu)$$

$$\equiv \exp\left(\tfrac{1}{2} \log \nu - i\nu\pi\right) \left[\exp\left(\nu \log \frac{\nu}{ez} - \frac{i\pi}{2}\right) - \exp\left(-\nu \log \frac{\nu}{ez}\right) \right] .$$

If this is to vanish, then

$$(7.6)\qquad 2\nu \log \frac{\nu}{ez} = \log 1 + \frac{i\pi}{2} = (2n + \tfrac{1}{2})\, i\pi$$

for integral values of n. Separating this into real and imaginary parts, we obtain

$$(7.7)\qquad \rho \cos \theta \log \frac{\rho}{er} - \rho(\theta - \phi) \sin \theta = 0$$

and

$$(7.8)\qquad 2\rho \left[\sin \theta \log \frac{\rho}{er} + (\theta - \phi) \cos \theta \right] = (2n + \tfrac{1}{2})\, \pi.$$

From (7.7) it follows that $\cos \theta \to 0$ as $\rho \to \infty$. Then

$$\alpha = \rho \cos \theta \sim \frac{\rho(\frac{\pi}{2} - \phi)}{\log \frac{\rho}{er}} ,$$

and making use of the next results we can localize the large zeros as follows.

Theorem 7.1. Let $H_\nu^{1}(z) = 0$ with $0 < \theta < \pi$. If ρ is sufficiently large, the zeros lie near the curve

$$\alpha = \frac{\rho(\frac{\pi}{2} - \phi)}{\log \frac{\rho}{er}}, \qquad \beta = \rho$$

where ρ satisfies

$$2\rho\left[\log \frac{\rho}{er} - \frac{(\frac{\pi}{2} - \phi)^2}{\log \frac{\rho}{er}}\right] = (2n + \tfrac{1}{2})\,\pi .$$

As an alternative, we may say that, for sufficiently large integers n, real and imaginary parts of the zeros of $H_\nu^{(1)}(z)$ are given by

$$\alpha = \pi(\tfrac{\pi}{2} - \phi)\,(n + \tfrac{1}{4})\left[\log(n + \tfrac{1}{4})\frac{\pi}{er}\right]^{-2}\left\{1 + \epsilon_n\right\}$$

$$\beta = \pi(n + \tfrac{1}{4})\left[\log(n + \tfrac{1}{4})\frac{\pi}{er}\right]^{-1}\left\{1 + \delta_n\right\}$$

where $\epsilon_n\ \delta_n = O\left[\frac{1}{\log n}\right]$.

In order to show that the large zeros of H_ν^{1} approximate those of $B(\nu)$, we shall prove that $|B(\nu)| > |D(\nu)|$ on some circle centered at a zero of $B(\nu)$ and then apply Rouché's theorem.

Let $B(\nu_o) = 0$, and consider the circle C: $\nu - \nu_o = \epsilon e^{i\omega}$, where $\nu_o = \rho e^{i\theta}$ and $\epsilon > 0$. Making use of (7.7) and (7.8), we can show that

$$\left|\sqrt{2\pi}\ B(\nu)\ e^{-\frac{1}{2}\log \nu + i\nu\pi}\right| \sim \left|-i\ e^{\nu \log \nu/ez} + e^{-\nu \log \nu/ez}\right|_C$$

$$\sim \left|-i\ e^{a + bi} + e^{-a - bi}\right|_C$$

$$= \sqrt{2}\ (\cosh 2a + \sin 2b)^{1/2}$$

where

$$2a = 2\epsilon\left[\cos \omega \cdot \log \frac{\rho}{r} - (\tfrac{\pi}{2} - \phi)\sin \omega\right]$$

and

$$2b = (2n - \frac{1}{2})\,\pi + 2\epsilon \sin\omega \cdot \log\frac{\rho}{er} + 2\epsilon(\frac{\pi}{2} - \phi)\cos\omega.$$

Now we get a bound for $|D(\nu)|$.

$$|D_1(\nu)| = \left|\frac{\nu z^{-\nu+2}}{\Gamma(-\nu+2)} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(k+1)!\,(-\nu+2)_k}\right|$$

$$< \left|\frac{z^{-\nu+2}}{(-\nu+1)\Gamma(-\nu)}\right| e^{r^2},$$

and

$$|D_2(\nu)| = \left|\frac{\nu e^{-i\nu\pi} z^{\nu+2}}{\Gamma(\nu+2)} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(k+1)!\,(\nu+2)_k}\right|$$

$$< \left|\frac{e^{-i\nu\pi} z^{\nu+2}}{(\nu-1)\Gamma(\nu)}\right| e^{r^2}$$

for $\alpha > 0$. Then

$$|D(\nu)|_C < \frac{r^2 e^{r^2}}{|\nu-1|}\left(\left|\frac{z^{-\nu}}{\Gamma(-\nu)}\right| + \left|\frac{e^{-i\nu\pi} z^{\nu}}{\Gamma(\nu)}\right|\right)_C .$$

As in the case of $B(\nu)$, we have

$$\left|\sqrt{2\pi}\, D(\nu)\, e^{-1/2 \log \nu + i\nu\pi}\right|_C < \frac{\sqrt{2} r^2 e^{r^2}}{\rho}\cosh a .$$

Then to show that $|B(\nu)| > |D(\nu)|$ on C, it suffices to show that, if $\mathbf{c} = 2r^4 e^{2r^2}$, $\cosh 2a + \sin 2b > \frac{c}{\rho^2}\cosh^2 a$ or $\left(1 - \frac{c}{\rho^2}\right)\cosh^2 a + \sinh^2 a > -\sin 2b$ where

$$-\sin 2b = \cos\left[2\epsilon \sin\omega \cdot \log\frac{\rho}{er} + 2\epsilon(\frac{\pi}{2} - \phi)\cos\omega\right] .$$

Choose

$$(7.9) \qquad \epsilon = \frac{1}{2 \log \frac{\rho}{er}} .$$

Then $1 - \frac{c}{\rho^2} > -\sin 2b$ if $|\sin\omega| \geq \frac{\sqrt{2}}{2}$ say, for ρ large enough. Now let ω take on the remaining values, i.e., $|\cos\omega| > \frac{\sqrt{2}}{2}$. Then

$$(1 - \frac{c}{\rho^2}) \cosh^2 a > (1 - \frac{c}{\rho^2}) \cosh^2 (\tfrac{1}{2}) > 1$$

for ρ sufficiently large. Consequently $|B(\nu)| > |D(\nu)|$ on C, and Rouché's theorem tells us that the function $B(\nu) + D(\nu)$ has as many zeros in C as does $B(\nu)$. A similar argument would establish the proximity of the zeros of $H_\nu^1 = A(\nu) + D(\nu)$ to those of $B(\nu)$.

We note that this proximity has already been determined quantitatively by our choice (7.9) of ϵ, if we ignore the slight perturbation introduced by the approximation $B(\nu)$ of $A(\nu)$.

8. The zeros of the Wronskian

If the sphere is not a perfect conductor, the role of $H_\nu^1(ka)$ is taken by the function

$$(8.1) \qquad W_\nu(ka, k'a) = J_\nu(k'a) \frac{d}{da} H_\nu^1(ka) - H_\nu^1(ka) \frac{d}{da} J_\nu(k'a)$$

where a is the radius of the sphere and k and k' are the wave numbers of the incident and diffracted waves, respectively. By applying similar techniques to W_ν as to H_ν, we shall obtain similar results. The first of these guarantees the existence of an infinite number of zeros.

For ν large, since the order of the entire function $\frac{1}{\Gamma(\nu)}$ is not greater than one, we have the following relations for any positive ϵ:

$$J_\nu(z) \sim \frac{(z/2)}{\Gamma(\nu+1)} = O(e^{\rho^{1+\epsilon}})$$

and

$$\frac{d}{dz} J_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z) = O(e^{\rho^{1+\epsilon}}).$$

Moreover, from Section 2, we know that if $y > 0$, $H_\nu^1(z) = O(e^{\rho^{1+\epsilon}})$ and therefore

$$\frac{d}{dz} H_\nu^1(z) = \frac{\nu}{z} H_\nu^1(z) - H_{\nu+1}^1(z) = O(e^{\rho^{1+\epsilon}}).$$

Consequently, the order of growth of the entire function W_ν is not greater than unity if $\operatorname{Im} k > 0$. This is true also if $\operatorname{Re} k > 0$; however, the proof, which follows, is more complicated.

From equations (7.1) to (7.3), if $\alpha \leq 0$, we can easily obtain

$$W_\nu(2ka,\ 2k'a) \sim \frac{(k'a)\Gamma(-\nu)}{i\pi a} \begin{vmatrix} a & \dfrac{(ka)^{-\nu}}{\Gamma(-\nu)} + \dfrac{e^{-i\nu\pi}(ka)^{\nu}}{\Gamma(\nu)} \\ 1 & -\dfrac{(ka)^{-\nu}}{a\Gamma(-\nu)} + \dfrac{e^{-i\nu\pi}(ka)^{\nu}}{a\Gamma(\nu)} \end{vmatrix}$$

whence $W_\nu = O(e^{\rho^{1+\epsilon}})$ for $\alpha \leq 0$. Now if $\alpha > 0$ and $x > 0$, then (see [1], p. 27)

$$|H_\nu^1(z)| = |c(\nu)\ I(\nu)|$$

where

$$c(\nu) = O(e^{\rho^{1+\epsilon}})$$

and

$$|I(\nu)| = \left| \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu - 1/2}t \exp\left[it(-\nu + \frac{1}{2}) - 2z \operatorname{ctg} t\right]}{\sin^{2\nu+1}t}\, dt \right|$$

$$\leq \int_0^{\frac{\pi}{2}} \frac{\exp\left[\beta t - 2x \operatorname{ctg} t\right]}{\sin^{2\alpha+1}t}\, dt .$$

$$< e^{\rho\pi/2} \int_0^\infty e^{-2xs}(s^2+1)^\alpha\, ds$$

$$< e^{\rho\pi/2}\, 2^\alpha \left[\int_0^1 e^{-2xs}\, ds + \int_1^\infty e^{-2xs}\, s^{2\alpha} ds\right]$$

where we let $\operatorname{ctg} t = s$. The bracketed term is less than $\frac{1}{2x} + \frac{\Gamma(2\alpha+1)}{(2x)^{2\alpha+1}}$. Hence, when $\alpha > 0$ and $x > 0$, $H_\nu^1(z)$ and its derivative equal $O(e^{\rho^{1+\epsilon}})$. Consequently $W_\nu = O(e^{\rho^{1+\epsilon}})$ for all ν; i.e., the order of growth of W_ν is ≤ 1.

Little now remains to prove the next theorem.

Theorem 8.1. If $-\frac{\pi}{2} < \arg ka < \pi$ and $-\pi < \arg k'a < \pi$, then there are an infinite number of zeros of $W_\nu(ka, k'a)$.

Proof: The function $W_\nu W_{-\nu}$ is or order ≤ 1. Since it is an even function of ν, it is an entire function of ν^2, with respect to which its order is $\leq \frac{1}{2}$. Then it must have an infinite number of zeros, provided we can show it is not a polynomial.

This may be done by considering the behavior of $W_\nu W_{-\nu}$ along the β axis by using (8.2) which follows. The same proof would also indicate that $\left|W_{i\beta}-1\right| \sim \frac{a^2}{\beta}\left|k^2 - k'^2\right| e^{\beta(\pi - 2\phi)}$ for large positive β, whence the order of W_ν is exactly 1, since we know it to be greater than 1. Finally, we infer that $W_{i\beta} \neq 0$ for large β.

To find the distribution of the large zeros, we use definition (7.1) for H_ν^1. If we let $ka = z$ and $k'a = z' = r'e^{i\phi'}$, then for $|\phi| < \pi$, $|\phi'| < \pi$, and ν not an integer, we have for large ν

$$W_\nu(2z,2z') = \frac{1}{ia \sin \nu\pi} \begin{vmatrix} J_\nu(2z') & J_{-\nu}(2z) - e^{-i\nu\pi} J_\nu(2z) \\ z'J'_\nu(2z') & zJ'_{-\nu}(2z) - ze^{-i\nu\pi} J'_\nu(2z) \end{vmatrix}$$

$$\sim \frac{1}{2ia \sin \nu\pi} \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$

where

$$A_{11} = \frac{z'^{\nu}}{\Gamma(\nu+1)} \left(1 - \frac{z'^2}{\nu+1}\right),$$

$$A_{21} = \frac{z'^{\nu}}{\Gamma(\nu+1)} \left(\nu - \frac{\nu+2}{\nu+1} z'^2\right),$$

$$A_{12} = \frac{z^{-\nu}}{\Gamma(-\nu+1)} \left(1 - \frac{z^2}{-\nu+1}\right) - \frac{e^{-i\nu\pi} z^{\nu}}{\Gamma(\nu+1)} \left(1 - \frac{z^2}{\nu+1}\right),$$

and

$$A_{22} = -\frac{z^{-\nu}}{\Gamma(-\nu+1)} \left(\nu + \frac{\nu-2}{\nu-1} z^2\right) - \frac{e^{-i\nu\pi} z^{\nu}}{\Gamma(\nu+1)} \left(\nu - \frac{\nu+2}{\nu+1} z^2\right).$$

Using some elementary properties of the gamma function and the fact that $\log(\nu+2) \sim \log \nu + \frac{2}{\nu}$, we obtain

$$(8.2) \qquad -ia\pi \left(\frac{z}{z'}\right)^{\nu} W_{\nu}(2z, 2z') \sim 1 + \frac{z^{2\nu} e^{-i\nu\pi} \Gamma(-\nu)}{\Gamma(\nu+2)} (z^2 - z'^2)$$

$$\sim 1 + \exp\left[\frac{i\pi}{2} - 2\nu \log \frac{\nu}{ez} - \log \frac{\nu}{z^2 - z'^2}\right]$$

with the aid of Stirling's formula. Here we imposed the restriction $0 < \theta < \pi$ on ν. For the above expression to vanish, we must have

$$2\nu \log \frac{\nu}{ez} + \log \frac{\nu}{z^2 - z'^2} = (2n - \tfrac{1}{2}) i\pi$$

for integral values of n. Separating these terms into their real and imaginary components, we find as in the preceding section that θ approaches $\frac{\pi}{2}$ as ρ goes to infinity. Consequently, we obtain

Theorem 8.2. Let $W_{\nu}(2z, 2z') = 0$ with $0 < \theta < \pi$, $-\pi < \phi < \pi$, $-\pi < \phi' < \pi$. If ρ is sufficiently large, then $\theta \sim \frac{\pi}{2}$,

$$2\rho \cos\theta \log \frac{\rho}{er} - 2\rho\left(\frac{\pi}{2} - \phi\right) + \log \frac{\rho^2}{|z^2 - z'^2|} \sim 0,$$

and

$$2\rho \log \frac{\rho}{er} + 2\rho\left(\frac{\pi}{2} - \phi\right) \cos\theta - \arg(z^2 - z'^2) \sim (2n - \tfrac{1}{2})\pi .$$

Thus far, W_ν and H_ν^1 have had parallel treatments of their large zeros. If we consider large values of z instead, by virtue of the inequalities obtained in Section 3 we know that $H_\nu^1(z) \neq 0$ for rather general conditions. We can prove that $W_\nu(z,z')$ likewise does not vanish, but lacking any corresponding inequalities, we must do this directly. We let $x' = z' = x' + iy'$.

Theorem 8.3. Let both $|z|$ and $|z'|$ be large compared to 1 and $|\nu|$; $x \geq 0$, $y \geq 0$, $x' \geq 0$, and $y' > \frac{\beta\pi}{2}$. Then $W_\nu \neq 0$.

Proof:

$$aW_\nu = \begin{vmatrix} J_\nu(z') & H_\nu^1(z) \\ z'J_\nu'(z') & z\,H_\nu^{1\prime}(z) \end{vmatrix}$$

$$\sim \frac{2}{\pi} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \begin{vmatrix} z'^{-1/2}\cos(z' - \frac{\nu\pi}{2} - \frac{\pi}{4}) & z^{-1/2} \\ -z'^{1/2}\sin(z' - \frac{\nu\pi}{2} - \frac{\pi}{4}) & iz^{1/2} \end{vmatrix}$$

or

$$\frac{a\pi}{2}(z\,z')^{1/2}\, e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} W_\nu \sim iz\cos(A+iB) + z'\sin(A+iB)$$

$$= R + i\,I$$

where

$$A = x' - \frac{\alpha\pi}{2} - \frac{\pi}{4}, \quad B = y' - \frac{\beta\pi}{2} > 0$$

and

$$R = \sin A(x\sinh B + x'\cosh B) - \cos A(y\cosh B + y'\sinh B)$$

$$I = \sin A(y\sinh B + y'\cosh B) + \cos A(x\cosh B + x'\sinh B).$$

Since the determinant of this linear system,

$$\tfrac{1}{2}(x^2 + x'^2 + y^2 + y'^2)\sinh 2B + \tfrac{1}{2}(xx' + yy')\cosh 2B,$$

never vanishes, neither does W_ν.

9. Special cases

In the preceding section we let ν or z and z' assume large values. We shall now eliminate this restriction. Lacking any of the integral representations such as were available for H_ν^1, we must content ourselves with much less general results. We shall for the most part limit the variables to real and imaginary values. This will enable us to apply elementary considerations, particularly the fact that the Wronskian of linearly independent solutions of Bessel's equation is never zero. The first theorem is a good illustration of this.

Theorem 9.1. If z is real, z' is positive or pure imaginary, and ν is real, then $W_\nu(z,z') \neq 0$.

Proof: Using the identity $H_\nu^1(z) = J_\nu(z) + i\, Y_\nu(z)$, we can rewrite the definition (8.1) of W_ν as

$$W = \left[J_\nu(k'a)\, \frac{d}{da} J_\nu(ka) \;-\; J_\nu(ka)\, \frac{d}{da} J_\nu(k'a)\right]$$

$$+ i \left[J_\nu(k'a)\, \frac{d}{da} Y_\nu(ka) \;-\; Y_\nu(ka)\, \frac{d}{da} J_\nu(k'a)\right] .$$

Each term in brackets is real, except for a common factor when k' is imaginary as is evident from the series representation for J_ν. If these terms vanish simultaneously, then

$$J_\nu(ka)\, \frac{d}{da} Y_\nu(ka) \;-\; Y_\nu(ka)\, \frac{d}{da} J_\nu(ka) = 0.$$

But this is impossible. Recalling that $z = ka$ and $z' = k'a$, we then obtain the statement of the theorem.

Theorem 9.2. If $\arg z = \frac{\pi}{2}$ and $\arg z' = 0$ or $\frac{\pi}{2}$, then the zeros of $W_\nu(z,z')$ occur in conjugate pairs. If in addition $\nu = i\beta \neq 0$, then $W_\nu(z,z') \neq 0$.

Proof: Taking conjugates of both sides of (8.1), when $\arg k = \frac{\pi}{2}$, $\arg k' = 0$ or $\frac{\pi}{2}$, and $c = -e^{-i\pi\bar{\nu}}$ or -1 respectively, we find

$$\overline{W}_\nu = c\left[J_{\overline{\nu}}(k'a)\,\frac{d}{da}\,H^1_\nu(ka) \;-\; H^1_\nu(ka)\,\frac{d}{da}\,J_{\overline{\nu}}(k'a)\right] \;=\; cW_{\overline{\nu}}.$$

This proves the first part of the theorem. Now when $\nu = i\beta \neq 0$,

$$\overline{W}_\nu \;=\; ce^{i\pi\nu}\left[J_{-\nu}(k'a)\,\frac{d}{da}\,H^1_\nu(ka) \;-\; H^1_\nu(ka)\,\frac{d}{da}\,J_{-\nu}(k'a)\right].$$

Then if $\overline{W}_\nu$ (and hence W_ν) vanishes, the determinant

$$J_\nu\,\frac{d}{da}\,J_{-\nu} \;-\; J_{-\nu}\,\frac{d}{da}\,J_\nu \;=\; 0,$$

where the argument is k'a. But this is impossible with our choice of ν.

This takes care of the case in which $z(=k'a = x' + iy' = r'e^{i\phi'})$ is real or pure imaginary. We will now treat the remaining values of z'. This will permit us to consider, instead of W_ν, a function which is simpler in many respects. The results will be valid also for real ν.

Suppose $\arg ka = \frac{\pi}{2}$; $\nu = i\beta$ or α; and k' neither real nor imaginary, i.e., $x'y' \neq 0$. Then

$$\overline{W}_\nu \;=\; c'\left[J_{\overline{\nu}}(\overline{k}'a)\,\frac{d}{da}\,H^1_\nu(ka) \;-\; H^1_\nu(ka)\,\frac{d}{da}\,J_{\overline{\nu}}(\overline{k}'a)\right],$$

where $c' = -1$ or $-e^{i\pi\nu}$ according as $\nu = i\beta$ or α respectively. Consequently W_ν (and hence $\overline{W}_\nu$) vanishes only with the determinant $\Delta(a)$, where

$$\Delta(a) \;=\; J_\nu(k'a)\,\frac{d}{da}\,J_{\overline{\nu}}(\overline{k}'a) \;-\; J_{\overline{\nu}}(\overline{k}'a)\,\frac{d}{da}\,J_\nu(k'a). \tag{9.1}$$

Now we apply the following formula ([1], p. 20):

$$a\Delta(a) \;=\; (k'^2 - \overline{k}'^2)\int t J_\nu(k't)\,J_{\overline{\nu}}(\overline{k}'t)\,dt, \tag{9.2}$$

whence

$$\frac{d}{da}\left[\frac{a\Delta(a)}{i}\right] \;=\; \frac{4\,x'y'}{a}\left|J_\nu(k'a)\right|^2. \tag{9.3}$$

To show that $\Delta(a)$ never vanishes, it suffices to show that either the derivative or $\lim_{a\to 0+}\frac{a\Delta(a)}{i}$ is positive, and that the other is non-negative. We shall do this on the assumption $x'y' > 0$.

Suppose first that $\nu = i\beta$. Then from (9.1),

$$\lim_{a\to 0+}\frac{a\Delta(a)}{i} \;=\; \frac{2\nu}{i\,\Gamma(\nu+1)\,\Gamma(-\nu+1)}\left(\frac{k'a}{2}\right)^{\nu}\left(\frac{\overline{k}'a}{2}\right)^{-\nu} = -\frac{2}{\pi}\sinh\beta\pi\; e^{-2\beta\phi'}.$$

This is positive if $\beta < 0$. Moreover, the derivative in (9.3) is obviously non-negative, since the radius a is positive, as always.

Now let $\nu = \alpha$. We note that if $W_\alpha = 0$ then $J_\alpha(k'a) \neq 0$, for otherwise $H_\alpha^1(ka) = 0$, contradicting Theorem 3.1. Then the derivative in (9.3) is positive. Furthermore, when $\alpha > 0$, from (9.1) we find that the limit of $\frac{a\Delta(a)}{i}$ vanishes. Then $\Delta(a) \neq 0$ for $a > 0$, which implies the same for W_ν.

A similar argument can be made if $x'y' < 0$, which proves

Theorem 9.3. Let $\arg z = \frac{\pi}{2}$, $x'y' \neq 0$, and $\nu = i\beta$ or $\nu = \alpha$.

Then $W_\nu(z,z') \neq 0$ whenever $\beta x'y' < 0$ or $\alpha > 0$ respectively.

Acknowledgement

Work on this report has been aided by the U.S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey, which released time for Mr. Kotin to work on the problem.

APPENDIX

Survey of results about the zeros of $H_\nu^1(x)$, $x > 0$

Let x be a fixed positive number. Let $\nu = \alpha + i\beta$ be a complex number and let $H_\nu^{(1)}(x)$ be the Hankel function of the first kind and of order ν with argument x. We have:

(1) The function $H_\nu^{(1)}(x)$ is an entire function of ν which vanishes for infinitely many values of $\nu = \pm \nu_k$, $k = 1,2,3 \ldots$.

(2) $$e^{i\nu\pi/2} H_\nu^{(1)}(x) = H_o^{(1)}(x) \prod_{k=1} \left(1 - \frac{\nu^2}{\nu_k^2}\right),$$

where the ν_k depend on x.

(3) For values $|\nu|$ which are sufficiently large compared to 1 and to x, the zeros of $H_\nu^{(1)}(x)$ are given by $\nu_n = \pm (\alpha_n + i\beta_n)$ where

$$\alpha_n = \frac{\pi^2}{2} \left(n + \tfrac{1}{4}\right) \left\{\log \left[\left(n + \tfrac{1}{4}\right) \frac{\pi}{ex}\right]\right\}^{-2} \left[1 + O\left(\frac{\log \log n}{\log n}\right)\right]$$

$$\beta_n = \pi \left(n + \tfrac{1}{4}\right) \left\{\log \left[\left(n + \tfrac{1}{4}\right) \frac{\pi}{ex}\right]\right\}^{-1} \left[1 + O\left(\frac{\log \log n}{\log n}\right)\right]$$

and where n is a (large) positive integer.

This shows that the distance of two consecutive zeros of $H_\nu^{(1)}(x)$ tends towards zero, and that the argument of the zeros tends towards $\pi/2$, although their real part tends towards infinity. Therefore, the behavior of the zeros as derived from the well-known formulas of van der Pol and Bremmer is not the final one. According to van der Pol and Bremmer, we have for large x (that is, for $x \gg 1$), the approximate expression

$$\nu_n \sim x + x^{1/3} \tau_n$$

for the n-th zero ν_n of $H_\nu^{(1)}(x)$, where for large n

$$\tau_n \sim \tfrac{1}{2}\left[3\pi\left(n + \tfrac{3}{4}\right)\right]^{2/3} e^{i\pi/3}$$

and where

$$\tau_o = 1.842\ e^{i\pi/3}$$

$$\tau_1 = 3.240\ e^{i\pi/3}$$

and so on.

(4) For the zeros ν_k of $H_\nu^{(1)}(x)$, with k = 1,2,3 ..., we have

$$\operatorname{Re} \sum_{k=1}^{\infty} \frac{1}{\nu_k^2} = - \frac{\int_0^\infty t^2 K_o(2x \sinh t)\, dt}{\int_0^\infty K_o(2x \sinh t)\, dt}$$

where K_o denotes the modified Hankel function of order zero.

(5) The following inequalities hold: If $\nu = \alpha + i\beta$ and $H_\nu^{(1)}(a\nu) = 0$, where a is real and positive and $\beta > 0$, then $a < 1$.

If $\nu = \alpha + i\beta$ and $H_\nu^{(1)}(x) = 0$ where $x > 0$, then, $\alpha\beta > 0$,

$$|\alpha| > x - \sqrt{x/2}$$

and

$$|\alpha| < \tfrac{1}{2} \implies x \leq \beta \tan \alpha\pi + \tfrac{1}{2}$$

and

$$|\alpha| = \tfrac{1}{2} \implies |\beta| > \tfrac{x}{2}$$

and

$$x \to 0 \implies \alpha \to 0,\ \alpha\beta \to 0 .$$

(6) If $\nu = \alpha + i\beta$ and $H_\nu^{(1)}(x) = 0$, then α, β may be considered as functions of the positive real variable x. We have

$$\frac{d\alpha}{dx} > 0 \text{ whenever } \alpha \geq \tfrac{1}{8} \text{ or } x \geq \tfrac{1}{4} .$$

$$\frac{d\beta}{dx} > 0 \text{ whenever } \alpha > 0 \text{ and } x > 0.$$

$$\lim_{x \to 0} \left| \frac{d\alpha}{dx} + i \frac{d\beta}{dx} \right| = \infty .$$

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